Journal of Computational and Applied Mathematics 30 (1990) 235–246
North-Holland

Solving nonlinear systems of equations with only one nonlinear variable

Y.-Q. SHEN and T.J. YPMA
Department of Mathematics, Western Washington University, Bellingham, WA 98225, USA

Received 6 June 1989
Revised 8 October 1989

Abstract: We describe a method for solving systems of $N+1$ nonlinear equations in $N+1$ unknowns $y \in \mathbb{R}$ and $z \in \mathbb{R}^N$ of the form $A(y)z + b(y) = 0$, where the $(N+1)\times N$ matrix $A(y)$ and vector $b(y)$ are functions of y alone. Such equations arise in minimax approximation. We reduce the problem to one equation in y only. An efficient quadratically convergent numerical technique based on Newton's method in one variable is used to solve this equation. Computational details and results are provided, and two generalizations are discussed.

Keywords: Nonlinear equations, Newton's method, minimax approximation, Remez algorithm.

1. Introduction

Systems of $N+1$ nonlinear equations in $N+1$ unknowns $y \in \mathbb{R}$ and $z \in \mathbb{R}^N$ of the special form

$$A(y)z + b(y) = 0, \tag{1}$$

where the $(N+1)\times N$ matrix $A(y)$ and vector $b(y)$ are functions of the scalar variable y only, arise in a variety of contexts in approximation theory [2,7]. A number of techniques for solving such problems, adapted to the particular situation concerned, have been described in the approximation theory literature [4,7,9,13]. For example, the Fraser–Hart iteration [4,9] has long been popular in the context of the Remez algorithm for minimax approximation by rational functions.

More recently it has been noted [5,7] that for fixed values of y (1) is an overdetermined system of linear equations in z; this leads to the following proposal. Select a value for y, distinguish one of the component equations of (1), solve the remaining N component equations of (1) for z, and determine whether the selected y and computed z satisfy the distinguished equation. If not, then somehow select a better value of y. In essence this reduces the problem to solving a single nonlinear equation in y only, since the computed z is clearly a function of y in this process. Any convenient technique for solving one nonlinear equation in one unknown could be used to solve this equation, with the advantage that only estimates of y are required to initiate the iteration. Several open questions concerning this approach, such as the possible singularity of the selected $N \times N$ submatrix of $A(y)$ used and the differentiability of z as a function of y, were raised in [7]. This approach is closely related to those described in [2,6,13].

In this paper we propose an alternative method to solve (1). It shares with the method outlined above the fact that the problem is reduced to solving a single nonlinear equation in y only, but z is never computed until the correct value of y is found, and the issues noted in [7] are resolved. In the next section we show how (1) can be reduced to a single nonlinear equation in y without distinguishing any particular component equation. We derive the differentiability properties of the function defining the resulting equation. In the subsequent section we show how this function and its derivative can be evaluated efficiently, and hence that Newton's method in one variable can be used effectively to solve the nonlinear equation. No value of z needs to be computed until the correct value of y has been found. The rate of convergence is quadratic. Once y has been found, the corresponding z is readily calculated. Some examples to illustrate the technique and compare it with other methods are presented in Section 4. Finally we discuss the extension of this technique to the case $y \in \mathbb{R}^m$ $(m > 1)$ and its application to problems of the form

$$A(y)G(v) + b(y) = 0,$$

where $y \in \mathbb{R}^m$ $(m \geqslant 1)$ and $G: \mathbb{R}^M \to \mathbb{R}^N$, with possibly $M \neq N$.

The technique of this paper may be regarded as an example of the general approach of tearing, which has appeared in the economics literature as a method for solving sparse systems and certain classes of dynamic models—see, for example, [8] for a summary and related references.

Throughout this paper we assume that

$$\operatorname{rank}(A(y)) = N$$

for all y of interest. However we do not assume that any particular $N \times N$ submatrix of $A(y)$ remains nonsingular for all relevant values of y.

2. Analysis

The technique to be presented in this paper is based on the following observation.

Theorem 1. *Let A be an $(N+1) \times N$ matrix with rank N. Then $x \in \mathbb{R}^N$ satisfies*

$$Ax + b = 0 \tag{2}$$

for given $b \in \mathbb{R}^{N+1}$ if and only if

$$x = -A^+ b \tag{3}$$

and

$$c^{\mathrm{T}} b = 0, \tag{4}$$

where $c \in \mathbb{R}^{N+1}$ satisfies

$$A^{\mathrm{T}} c = 0, \tag{5}$$

$$c^{\mathrm{T}} c = 1, \tag{6}$$

and A^+ is the Moore–Penrose pseudo-inverse of A.

Proof. Conditions (4)–(6) are simply conditions for the existence of a solution to (2): by elementary linear algebra [10] a solution x of (2) exists if and only if $-b \in \operatorname{range}(A)$, which

occurs if and only if b is orthogonal to null(A^T), which is equivalent to (4)–(6) with c a unit basis vector for the space null(A^T) which has dimension one by our rank assumption. Then (3) follows since by our rank assumption $-A^+b$ is the unique solution of $\min_x \| Ax + b \|_2$. □

Applying Theorem 1 to equation (1) we see that (1) is equivalent to finding y and z such that

$$z = -A(y)^+ b(y), \tag{7}$$

where y satisfies

$$c(y)^T b(y) = 0 \tag{8}$$

and $c(y)$ is such that

$$A(y)^T c(y) = 0, \tag{9}$$

$$c(y)^T c(y) = 1. \tag{10}$$

The idea is to solve (8) for y: this is a single equation in one unknown y to which in principle any standard technique can be applied. To every solution y of (8) there corresponds a z given by (7) such that (y, z) solves (1), hence all solutions of (1) can be obtained by finding all solutions of (8), while (8) has no solutions if and only if (1) has no solutions. In particular there is no need to estimate z in order to initiate an iterative method applied directly to (1); we need only iterate on y from an initial y estimate to solve (8). Moreover there is no need to compute any values of z until y satisfying (7) has been calculated.

We introduce the notation

$$f(y) = c(y)^T b(y). \tag{11}$$

In order to solve $f(y) = 0$ we shall wish to exploit continuity and differentiability properties of $f(y)$. Assume that both $A(y)$ and $b(y)$ are continuously differentiable with respect to y. We will show that $c(y)$ can be defined in such a way that $c(y)$ is also continuously differentiable with respect to y, and hence so is $f(y)$.

Recalling that rank($A(y)$) = N and hence that null($A(y)^T$) has dimension 1, while from (9) $c(y) \in$ null($A(y)^T$), it follows that (9)–(10) specify $c(y)$ uniquely, up to a choice of sign, as a unit basis vector for null($A(y)^T$). In order to discuss differentiability we require that the choice be specified uniquely. We therefore require that, if $c(y_1)$ is given for some y_1, then for all y_2 sufficiently near y_1 the sign of $c(y_2)$ is selected so that

$$c(y_1)^T c(y_2) > 0. \tag{12}$$

Condition (12) is imposed for analytical purposes only; in the numerical method we subsequently propose the selection of a sign for $c(y)$ will be shown to be irrelevant. We emphasize also that $c(y)$ need never be expressed analytically; we will demonstrate how to evaluate $c(y)$ efficiently numerically.

The following theorem establishes the existence of a continuously differentiable basis $c(y)$ for null($A(y)^T$), and gives an equation satisfied by $c'(y)$. Related results concerning the existence and Lipschitz continuity of a unique $c(y)$ are proved by different techniques in a different context in [12].

Theorem 2. *If* $\operatorname{rank}(A(y)) = N$ *and* $A(y)$ *is continuously differentiable for all* $y \in (\alpha, \beta)$, *then for all* $y \in (\alpha, \beta)$ *the function* $c(y)$ *given by* (9), (10) *and* (12) *is continuously differentiable and satisfies*

$$\begin{bmatrix} A(y)^{\mathrm{T}} \\ \hline c(y)^{\mathrm{T}} \end{bmatrix} c'(y) = \begin{bmatrix} -A'(y)^{\mathrm{T}} c(y) \\ \hline 0 \end{bmatrix}, \tag{13}$$

where $A'(y)$ *is the* $(N+1) \times N$ *matrix whose entries are the derivatives with respect to* y *of the entries of* $A(y)$.

Proof. Observe that if $c(y)$ is known to be differentiable, then (13) can be obtained by differentiating (9) and (10) with respect to y.

We first prove that $c(y)$ is continuous. Fix y and let y^+ be near y. Then $c(y^+)$ is uniquely defined. Observe that (9) may be used to write

$$\begin{aligned} A(y)^{\mathrm{T}}[c(y) - c(y^+)] &= -A(y)^{\mathrm{T}} c(y^+) = A(y^+)^{\mathrm{T}} c(y^+) - A(y)^{\mathrm{T}} c(y^+) \\ &= [A(y^+) - A(y)]^{\mathrm{T}} c(y^+). \end{aligned} \tag{14}$$

Using continuity of $A(y)$, taking limits on both sides of (14) as y^+ converges to y gives

$$A(y)^{\mathrm{T}} \Big[c(y) - \lim_{y^+ \to y} c(y^+) \Big] = 0.$$

Hence $c(y) - \lim_{y^+ \to y} c(y^+) \in \operatorname{null}(A(y)^{\mathrm{T}})$; but $c(y) \in \operatorname{null}(A(y)^{\mathrm{T}})$ and we have seen that $\operatorname{null}(A(y)^{\mathrm{T}})$ has dimension 1. Thus there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{y^+ \to y} c(y^+) = (1 - \gamma) c(y). \tag{15}$$

By (10) $c(y^+)^{\mathrm{T}} c(y^+) = 1 = c(y)^{\mathrm{T}} c(y)$, hence (15) implies that $\gamma = 0$ or $\gamma = 2$. But if $\gamma = 2$, then $\lim_{y^+ \to y} c(y^+) = -c(y)$ which contradicts $c(y^+)^{\mathrm{T}} c(y) > 0$ for all y^+ sufficiently near y. Thus $\gamma = 0$ and $\lim_{y^+ \to y} c(y^+) = c(y)$ as claimed.

The proof of differentiability follows similarly. For convenience write $y^+ = y + h$. Then by (14)

$$\lim_{h \to 0} A(y)^{\mathrm{T}} \frac{[c(y^+) - c(y)]}{h} = \lim_{h \to 0} \frac{[A(y) - A(y^+)]^{\mathrm{T}}}{h} c(y^+),$$

and hence

$$A(y)^{\mathrm{T}} \lim_{h \to 0} \frac{[c(y^+) - c(y)]}{h} = -A'(y)^{\mathrm{T}} c(y). \tag{16}$$

Using (10) it is easy to observe that

$$[c(y) + c(y^+)]^{\mathrm{T}} \frac{[c(y^+) - c(y)]}{h} = 0.$$

Letting y^+ converge to y we get

$$2c(y)^{\mathrm{T}} \lim_{h \to 0} \frac{[c(y^+) - c(y)]}{h} = 0. \tag{17}$$

Combining (16) and (17) gives

$$\begin{bmatrix} A(y)^{\mathrm{T}} \\ c(y)^{\mathrm{T}} \end{bmatrix} \lim_{h \to 0} \frac{[c(y^{+}) - c(y)]}{h} = \begin{bmatrix} -A'(y)^{\mathrm{T}} c(y) \\ 0 \end{bmatrix}. \tag{18}$$

But the matrix on the left in (18) is nonsingular, since by (9) $c(y)$ is orthogonal to the rows of $A(y)^{\mathrm{T}}$. Thus the limit $c'(y)$ in (18) exists and satisfies (13). Moreover all the terms in the expression for $c'(y)$ obtained by solving (13) are continuous, hence $c'(y)$ is continuous. □

In view of our earlier observation that $c(y)$ is a unit basis vector for null$(A(y)^{\mathrm{T}})$, this theorem reveals the existence of a continuously differentiable basis for this space. Higher-order derivatives of $c(y)$ can be obtained in a similar way provided that $A(y)$ is sufficiently often differentiable. These higher-order derivatives satisfy systems of linear equations whose matrix on the left is identical to that in the system (13).

3. Computation

We propose the use of Newton's method to solve (8) for y. Thus we compute

$$y_{i+1} = y_i - \frac{f(y_i)}{f'(y_i)}, \quad i = 0, 1, \ldots, \tag{19}$$

where $f(y)$ is defined by (11). Clearly evaluation of $f(y)$ and $f'(y)$ requires evaluation of $c(y)$ and $c'(y)$. We will show how (9)–(10) can be used to numerically determine $c(y)$ efficiently, whence (13) may be solved to find $c'(y)$. We need only factorize one $(N+1) \times (N+1)$ matrix to find both $c(y)$ and $c'(y)$. The techniques used here are closely related to those frequently used in homotopy (or continuation) methods [1,12,14].

First note that in the context of the Newton iteration (19) the choice of sign of $c(y_i)$ is irrelevant: (19) takes the form

$$y_{i+1} = y_i - \frac{c(y_i)^{\mathrm{T}} b(y_i)}{\left[c(y_i)^{\mathrm{T}} b'(y_i) + c'(y_i)^{\mathrm{T}} b(y_i)\right]} \tag{20}$$

and it is easy to see from equation (13) that if the sign of $c(y)$ is altered, then the sign of the unique solution $c'(y)$ of (13) alters correspondingly, and these effects cancel in (20). Accordingly we need only satisfy (9)–(10), hence we require only that $c(y)$ be a unit vector in null$(A(y)^{\mathrm{T}})$.

Assume for the moment that we have available a vector $a(y) \in \mathbb{R}^{N+1}$ such that

$$\begin{bmatrix} A(y)^{\mathrm{T}} \\ a(y)^{\mathrm{T}} \end{bmatrix} \tag{21}$$

is nonsingular. Then we compute the unique solution $\bar{c} \in \mathbb{R}^{N+1}$ of

$$\begin{bmatrix} A(y)^{\mathrm{T}} \\ a(y)^{\mathrm{T}} \end{bmatrix} \bar{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{22}$$

so that $\bar{c} \neq 0$ and $\bar{c} \in$ null$(A(y)^{\mathrm{T}})$, and set $c(y) = \bar{c} / \| \bar{c} \|_2$. In practice we have used the

LU-decomposition to factorize the matrix (21), but clearly any useful structure or sparsity present in $A(y)$ should be exploited in the solution of (22) by using appropriate methods to factor the matrix.

While $c'(y)$ can be computed directly from (13) once $c(y)$ is known, it is more efficient to use the matrix factorization already computed in solving (22). Our assumption that $\operatorname{rank}(A(y)^{\mathrm{T}}) = N$ implies that all solutions d of

$$A(y)^{\mathrm{T}} d = -A'(y)^{\mathrm{T}} c(y) \tag{23}$$

have the form

$$d = \bar{d} + \alpha c(y), \quad \alpha \in \mathbb{R},$$

where $\bar{d}$ is any particular solution of (23), since $c(y) \in \operatorname{null}(A(y)^{\mathrm{T}})$. Thus we compute a solution $\bar{d}$ of

$$\left[\frac{A(y)^{\mathrm{T}}}{a(y)^{\mathrm{T}}}\right]\bar{d} = \left[\frac{-A'(y)^{\mathrm{T}} c(y)}{0}\right],$$

using the previously computed factors of the matrix involved, then set $c'(y) = \bar{d} + \alpha c(y)$ where α is selected to satisfy the last equation of (13) namely $c(y)^{\mathrm{T}} c'(y) = 0$. Using (10) this leads to

$$c'(y) = \bar{d} - \left(c(y)^{\mathrm{T}} \bar{d}\right) c(y).$$

Higher-order derivatives of $c(y)$, if required, can be computed in a similar way since they satisfy systems of linear equations with the same matrix on the left as in (13).

It is similarly possible to compute z given by (7) efficiently once y which satisfies (8) has been obtained. Letting $a(y)$ be as before, we will have factorized (21) at the current value of y in evaluating $c(y)$ in order to check whether the residual $f(y)$ satisfies the imposed termination conditions. Hence the matrix

$$[A(y) \mid a(y)]$$

is nonsingular and has known factors—the transpose of the factors of (21). Thus the equation

$$[A(y) \mid a(y)]\left[\frac{s}{t}\right] = -b(y) \tag{24}$$

has a unique solution which can be cheaply computed. But since (8)–(10) are satisfied, we know by Theorem 1 that there exists z, given by (7), which satisfies (1). Thus we know that $(z, 0)$ is a solution of (24), from which it follows by uniqueness that $z = s$.

The following result is useful in selecting $a(y)$ for use in (21).

Lemma 3. *If A and c are as defined in Theorem* 1 *and satisfy* (5)–(6), *then for all $a \in \mathbb{R}^{N+1}$*

$$\det\left(\left[\frac{A^{\mathrm{T}}}{a^{\mathrm{T}}}\right]\right) = (a^{\mathrm{T}} c) \det\left(\left[\frac{A^{\mathrm{T}}}{c^{\mathrm{T}}}\right]\right). \tag{25}$$

Proof. Equations (5) and (6) combine to read

$$\left[\frac{A^{\mathrm{T}}}{c^{\mathrm{T}}}\right] c = e_{N+1} \tag{26}$$

where e_{N+1} is the last column of the $(N+1)\times(N+1)$ identity matrix. By analogy with the final steps of the proof of Theorem 2 the matrix in (26) is nonsingular, hence the system in (26) has the unique solution c. Now

$$\begin{bmatrix} A^{\mathrm{T}} \\ a^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} A^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix} + e_{N+1}(a-c)^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix}\left(I + \left(\begin{bmatrix} A^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix}^{-1} e_{N+1}\right)(a-c)^{\mathrm{T}}\right)$$

$$= \begin{bmatrix} A^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix}\left(I + c(a-c)^{\mathrm{T}}\right).$$

Recalling that $\det(I+uv^{\mathrm{T}}) = 1 + u^{\mathrm{T}}v$ we find

$$\det\left(\begin{bmatrix} A^{\mathrm{T}} \\ a^{\mathrm{T}} \end{bmatrix}\right) = \det\left(\begin{bmatrix} A^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix}\right)(1 + c^{\mathrm{T}}(a-c)),$$

from which the result follows. □

Thus all that is required for nonsingularity of the matrix in (21) is that

$$a(y)^{\mathrm{T}}c(y) \neq 0,$$

while an optimal choice of $a(y)$, in the sense of maximizing the determinant, maximizes $|a(y)^{\mathrm{T}}c(y)|$. If we restrict ourselves to $a(y)$ such that $a(y)^{\mathrm{T}}a(y) = 1$, then the optimal choice is clearly $a(y) = c(y)$. Since this is impractical in that the matrix in (21) is used to find $c(y)$, we seek a compromise. We propose to use

$$a(y_i) = c(y_{i-1})$$

in the course of the Newton iteration. When the iteration is converging and in its final stages, we expect y_i to be near y_{i-1}, hence by continuity we expect $c(y_{i-1})$ (and thus $a(y_i)$) to be near $c(y_i)$, so that $|a(y_i)^{\mathrm{T}}c(y_i)|$ will be large (near 1). In particular, if we have an accurate estimate of y to start the iteration, then this technique works well and the resulting matrix is reasonably well-conditioned. When further away from the solution the values y_i change more radically and there is no guarantee that our choice of $a(y_i)$ will be acceptable. To start the iteration an arbitrary choice of $a(y_0)$ must be made; if a random number generator is used to produce a vector $a(y_0)$, then the probability that the resulting matrix is nonsingular is very high. In practice we have used

$$a(y_0) = e_{N+1}$$

without encountering any difficulties.

Naturally if $A(y)$ has a useful structure or sparsity, then $a(y)$ should if possible be selected to preserve that property in the expanded matrix (21). In line with the previous discussion we suggest that in such cases $a(y_i)$ be selected as the unit projection of $c(y_{i-1})$ onto the space with the relevant characteristics.

Finally we note that Newton's method occasionally encounters difficulties in solving a single nonlinear equation, and it is tempting to replace it by a more robust method with guaranteed convergence. Unfortunately most such methods are based on bracketing techniques which rely on the continuity and sign of $f(y)$, and hence are absolutely dependent on the correct sign of $c(y)$ being selected at every step. While condition (12) may be imposed in some form to

determine the sign of $c(y)$, this is a major source of difficulties. A safeguarded Newton method in which we compute [3]

$$y_{i+1} = y_i - \frac{t_i f(y_i)}{f'(y_i)}, \quad t_i \in (0, 1],$$

where t_i is selected so that

$$|f(y_{i+1})| < (1 - \beta t_i)\,|f(y_i)|,$$

for some $\beta \in [0, \frac{1}{2}]$, again avoids the problem of selection of sign. It is also possible that the function $f(y)$ has a multiple zero; since Newton's method converges only linearly to such solutions of $f(y) = 0$ a modification is required to restore quadratic convergence. Relevant techniques, largely based on using higher-order derivatives of $f(y)$ and hence of $c(y)$, are described in [15].

4. Experiments

Consider the simple though contrived example with $N = 2$

$$\begin{pmatrix} y & 0 \\ 1 & 1 \\ 1+y & 1-y^2 \end{pmatrix} z + \begin{pmatrix} y \\ y \\ 1+y \end{pmatrix} = 0. \tag{27}$$

It is readily confirmed that this has exactly two solutions, namely $y = +1$ and $y = -1$, with $z^{\mathrm{T}} = (-1, 0)$ and $z^{\mathrm{T}} = (-1, 2)$, respectively. For $y \neq 0$ the rank of the matrix is 2. Thus for $y \neq 0$ our method may be used; for the purposes of analysis it can be shown that the equation $c(y)^{\mathrm{T}} b(y) = 0$ in this case reduces to

$$\frac{(1+y)(1-y)^2}{\sqrt{(1+y)^2 + (1-y^2)^2 + 1}} = 0.$$

Our numerical experience using the technique of the previous section coincides with what this analysis leads one to expect: we obtained quadratic convergence to the simple zero at $y = -1$ from appropriate starting points near -1, and linear convergence with factor $\frac{1}{2}$ to the solution of multiplicity 2 at $y = +1$ from other starting points (excluding 0). Modifying the Newton method to compensate for the higher-order solution at $+1$ resulted in quadratic convergence to that solution also. Although the matrix rank drops to 1 at $y = 0$ this caused no difficulty in our method provided we started at $y \neq 0$.

The method of [5,7] applied to (27) solves the bottom two equations for z and substitutes the result into the top equation. This fails at $y = 0, -1$ since then the bottom 2×2 submatrix is singular. Indeed, if started at $y = -1$ that method would fail without recognizing this as being a solution. For $y \neq 0, -1$ that method produces the nonlinear equation $(1-y)^2 = 0$, hence the solution of multiplicity 2 at $+1$ is the only solution found.

We also performed a series of experiments relevant to one context within which equations of the form (1) arise. Given a set of linearly independent basis functions $\phi_j(t)$ $(j = 0, 1, \ldots)$ we define

$$q_m(t) = \phi_0(t) + \sum_{j=1}^{m} z_j \phi_j(t), \qquad p_l(t) = \sum_{j=m+1}^{l+m+1} z_j \phi_{j-(m+1)}(t)$$

for given l and m, and consider approximating a given function $r(t)$, in the weighted minimax sense, by a rational approximation of the form $p_l(t)/q_m(t)$. Thus the coefficients z_j ($j = 1, \ldots, l + m + 1$) are to be determined. If the Remez exchange algorithm is applied to solve this problem, then at every iteration values t_i ($i = 0, \ldots, l + m + 1$) are generated, and the nonlinear system of equations

$$q_m(t_i)\left[(-1)^i \frac{y}{w(t_i)} - r(t_i)\right] + p_l(t_i) = 0, \quad i = 0, \ldots, l + m + 1, \tag{28}$$

must be solved for y and z_j ($j = 1, \ldots, l + m + 1$). Here $w(t_i)$ is a known positive weight.

Clearly the system of equations (28) can be written in the form (1). In our computational experiments we used the Chebyshev polynomials as basis functions; the weights were either uniformly set to 1 or else we used the Chebyshev weights $w(t_i) = 1/\sqrt{1 - t_i^2}$; the values t_i were generated by a random number generator and distributed uniformly over $(-1, 1)$. Various combinations of l and m, each between 0 and 5, were tried. A partial list of the functions r used includes various continuous functions without singularities on $[-1, +1]$, as well as $(\ln(1 + t))^2$, $1/(1 - t^2)$, $1/t$. In every case we obtained quadratic convergence to a solution of the problem from a wide range of starting points.

As an illustrative example we list the results for the case $l = 5$ and $m = 3$ in Table 1, starting from the points $y = 10$ and $y = -10$, using the Chebyshev weights and $r(t) = (\ln(1 + t))^2$. We list the first nine significant digits only of the iterates y_i and corresponding values of $f(y_i)$, and finally the computed solution vector z, computed using double precision Fortran. Quadratic convergence is apparent in both cases. By contrast our implementation of the Fraser–Hart algorithm following [4] required over 100 iterations to converge (linearly) to the same solution in both cases. We note that in practical use of the Remez algorithm a good initial estimate of the appropriate y value is generally available, so that we would not generally expect an improvement of this magnitude in that context.

In terms of significant computational cost, the technique proposed here requires the factorization of one $(N + 1) \times (N + 1)$ matrix, two subsequent forward and back substitutions, one matrix–vector product, and the evaluation of $A(y)$, $b(y)$, $A'(y)$, $b'(y)$ per iteration. Furthermore we require either that expressions for $A'(y)$ and $b'(y)$ be obtained analytically a priori (which is readily done in the case of rational minimax approximation when the Remez algorithm is used) or else that $A'(y_i)$ and $b'(y_i)$ be evaluated by some other means at the ith iteration of the algorithm—for which either numerical differences [3] or preferably automatic differentiation [11] may be used. By contrast the Fraser–Hart iteration requires only the factorization of one $(N + 1) \times (N + 1)$ matrix, one forward and back substitution, and the evaluation of $A(y)$ and $b(y)$ at every iteration. Thus the latter method is certainly cheaper per iteration, and the overall balance of costs depends largely on the cost of evaluating $A'(y)$ and $b'(y)$. It seems likely, however, that the cost of evaluating $A'(y)$ and $b'(y)$ will generally be comparable to the cost of evaluating $A(y)$ and $b(y)$, in which case the cost per iteration of the new method is about twice that of the Fraser–Hart method; the quadratic convergence of the new method compensates for this increased cost per iteration.

Our limited computational experience indicates that the method described is reasonably robust and efficient. It appears to be a useful alternative and possibly superior to existing methods in the context of approximation theory algorithms which give rise to problems of the class of interest in this paper. In particular the user need only provide an initial estimate of y,

Table 1
Solving (28); $l = 5$, $m = 3$; Chebyshev basis and weights; $r(t) = (\ln(1+t))^2$

i	y_i	$f(y_i)$	y_i	$f(y_i)$	y	z
0	$0.100000000 \cdot 10^{2}$	$0.125829141 \cdot 10^{-1}$	$-0.100000000 \cdot 10^{2}$	$-0.197876336 \cdot 10^{-1}$	$-0.393628345 \cdot 10^{0}$	$0.160752666 \cdot 10^{1}$
1	$-0.967496363 \cdot 10^{1}$	$-0.189931913 \cdot 10^{-1}$	$-0.194156452 \cdot 10^{1}$	$-0.249778144 \cdot 10^{-2}$		$0.917271671 \cdot 10^{0}$
2	$-0.186773238 \cdot 10^{1}$	$-0.236616356 \cdot 10^{-2}$	$-0.544026481 \cdot 10^{0}$	$-0.190914366 \cdot 10^{-3}$		$0.283419299 \cdot 10^{0}$
3	$-0.536933789 \cdot 10^{0}$	$-0.181251252 \cdot 10^{-3}$	$-0.404275341 \cdot 10^{0}$	$-0.123411128 \cdot 10^{-4}$		$0.560268434 \cdot 10^{0}$
4	$-0.403530587 \cdot 10^{0}$	$-0.114706882 \cdot 10^{-4}$	$-0.393722442 \cdot 10^{0}$	$-0.108088345 \cdot 10^{-6}$		$0.621221097 \cdot 10^{0}$
5	$-0.393710040 \cdot 10^{0}$	$-0.938412245 \cdot 10^{-7}$	$-0.393628352 \cdot 10^{0}$	$-0.890434399 \cdot 10^{-11}$		$0.577194312 \cdot 10^{0}$
6	$-0.393628350 \cdot 10^{0}$	$-0.671239601 \cdot 10^{-11}$	$-0.393628345 \cdot 10^{0}$	$-0.104783322 \cdot 10^{-15}$		$0.536351294 \cdot 10^{-1}$
7	$-0.393628345 \cdot 10^{0}$	$-0.233334919 \cdot 10^{-15}$				$0.493525863 \cdot 10^{-1}$
						$-0.320013592 \cdot 10^{-1}$

there is no need to distinguish any particular component of the initial system of equations, and we obtain quadratic convergence at relatively low cost.

5. Generalizations

The method described in this paper for scalar y can be extended to $y \in \mathbb{R}^m$ ($m \geqslant 1$). Though the general approach for $m > 1$ is similar to that above, the details of the analysis and implementation differ substantially. This extension is developed in [16].

A referee has pointed out that our method may also be applied to solve the more general problem

$$A(y)G(v) + b(y) = 0 \tag{29}$$

for the unknowns $y \in \mathbb{R}^m$ and $v \in \mathbb{R}^M$, where $G: \mathbb{R}^M \to \mathbb{R}^N$. Setting $z = G(v)$, the method of this paper (or its extension for $m > 1$) may be used to compute y for which (29) has a solution $z = G(v)$. The corresponding value of z having then been computed as indicated above, v can be found by solving the (possibly over- or underdetermined) system of N equations

$$G(v) - z = 0$$

for the M unknowns $v \in \mathbb{R}^M$, using an appropriate technique for this subproblem. Thus the original problem has been decomposed into two smaller subproblems. This observation highlights a particular advantage of the method developed in this paper, namely the separate computation of y and z.

References

[1] E. Allgower and K. Georg, Simplicial and continuation methods for approximating fixed points, *SIAM Rev.* **22** (1980) 28–85.

[2] I. Barrodale, F. Roberts and C. Hunt, Computing best l_p approximations by functions nonlinear in one parameter, *Computer J.* **13** (1970) 382–386.

[3] J.E. Dennis Jr. and R.B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations* (Prentice-Hall, Englewood Cliffs, NJ, 1983).

[4] C.B. Dunham, Convergence of the Fraser–Hart algorithm for rational Chebyshev approximation, *Math. Comp.* **29** (1974) 1078–1082.

[5] C.B. Dunham, Rationals with repeated poles, in: J. Mason and M.G. Cox, Eds., *Algorithms for Approximation* (Oxford Univ. Press, Oxford, 1987) 285–291.

[6] C.B. Dunham, Approximation with one (or few) parameters nonlinear, *J. Comput. Appl. Math.* **21** (1) (1988) 115–118.

[7] C.B. Dunham and C.Z. Zhu, Solving equations nonlinear in only one of $N+1$ variables, *ACM SIGNUM Newsletter* **23** (3 & 4) (1988) 2–3.

[8] P.G. Fisher and A.J. Hughes Hallett, Iterative techniques for solving simultaneous equation systems: a view from the economics literature, *J. Comput. Appl. Math.* **24** (1&2) (1988) 241–255.

[9] W. Fraser and J. Hart, On the computation of rational approximations to continuous functions, *Comm. ACM* **5** (1962) 401–402.

[10] G. Golub and C. Van Loan, *Matrix Computations* (North Oxford Academic, Oxford, 1983).

[11] A. Griewank, On automatic differentiation, Preprint MCS-P10-1088, Argonne National Laboratory, 1988.

[12] R. Menzel and H. Schwetlick, Zur Lösung parameterabhängiger nichtlinearer Gleichungen mit singulären Jacobi-Matrizen, *Numer. Math.* **30** (1978) 65–79.

[13] A. Ralston, Rational Chebyshev approximation, in: A. Ralston and H. Wilf, Eds., *Mathematical Methods for Digital Computers II* (Wiley, New York, 1967) 264–284.
[14] L.T. Watson, Numerical linear algebra aspects of globally convergent homotopy methods, *SIAM Rev.* **28** (1986) 529–546.
[15] T.J. Ypma, Finding a multiple zero by transformations and Newton-like methods, *SIAM Rev.* **25** (1983) 365–378.
[16] T.J. Ypma and Y.-Q. Shen, Solving $N + m$ nonlinear equations with only m nonlinear variables, *Computing*, to appear.